Numerical Solving of Third Degree Equation

Yotam Medini

November 19, 2009

Abstract

Dedicated to: *Hillel, Roey, Tomer and Yotam.*

Given the equation

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0 \qquad \text{(where } \alpha \neq 0\text{)} \tag{1}$$

we want to find a numerical solution. Dividing by α we get

$$x^3 + (\beta/\alpha)x^2 + (\gamma/\alpha)x + \delta/\alpha = 0.$$

If we put

$$b = \beta/\alpha \qquad c = \gamma/\alpha \qquad d = \delta/\alpha$$

we get a simplified equation:

$$p(x) = x^3 + bx^2 + cx + d = 0 \tag{2}$$

We want to find some large enough value M such that

$$p(-M) = -M^3 + bM^2 - cM + d \;<\; 0 \;<\; M^3 + bM^2 + cM + d = p(M). \tag{3}$$

We will show that if

$$M > \max(3|b|, \sqrt[2]{|3c|}, \sqrt[3]{|3d|}) \tag{4}$$

Then (3) will be true. We can simply pick

$$M = \max(3|b|, 2|c|, 2|d|) + 1 > \max\left(3|b|, \sqrt[2]{|3c|}, \sqrt[3]{|3d|}\right).$$

By the requirement (4) we are assured that

$$M/3 > |b| \qquad M^2/3 > |c| \qquad M^3/3 > |d|$$

Now

$$\begin{aligned} p(M) &= M^3 + bM^2 + cM + d \\ &\geq M^3 - |b|M^2 - |c|M - |d| \\ &= (M^3/3 - |b|M^2) + (M^3/3 - |c|M) + (M^3/3 - |d|) \\ &> 0. \end{aligned}$$

Similarly

$$\begin{aligned} p(-M) &= -M^3 + bM^2 - cM - d \\ &\leq -M^3 + |b|M^2 + |c|M + |d| \\ &= (|b|M^2 - M^3/3) + (|c|M - M^3/3) + (|d| - M^3/3) \\ &< 0. \end{aligned}$$